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Almost Periodic Solutions to $SO(2,1)$ σ-Model Field Theory

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We construct almost periodic conoidal wave solutions to $SO(2,1)$ σ-model field theory with the help of a periodic inverse problem suggested by Dote. The solutions can be explicitly constructed in terms of Riemann θ functions.

The inverse problem for nonlinear equations, both periodic and aperiodic, have been applied to several problems over the last few years. Detailed analyses are available about the periodic solutions for the nonlinear Schrödinger equation (Dubrovnin et al., 1976), the KdV proboem (Matveev, 1976), mixed nonlinear Schrödinger equation (Choudhary et al., 1985), and massive Thirring model (Dote, 1978). On the other hand some results about the three-dimensional equations have been obtained by Dabrovin (1975), Novikov (1974), and other Soviet authors. Here we study the periodic inverse problem for a completely different class of theory, the σ models, which are also known to be completely integrable. At present there exist various types of σ-model field theory—the $O(N)$ invariant whose special case is the sine-Gordon system. The sine-Gordon periodic problem has been thoroughly discussed by Forest and McLaughlin (1982). The other categories include the $SO(p, q)$ invariant σ-models (Eichemerr, 1982), not reducible to the sine-Gordon system, and lastly the Grassmanian σ-model theories (Zakrzewski, 1982). Of all these we thought that it is worthwhile to study the $SO(2,1)$ invariant σ-model system, whose Backlund transformation has been discussed by Chiniea (1981) and whose complete inverse problem on the infinite axis is done by Chowdhury and Mukherjee (1984). We essentially have followed the methodology put forward by Dote (Dote, 1978; Dobrovin, 1981) and his collaborators for our problem at hand.

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FORMULATION

The σ model under consideration is governed by the equations

$$Z_{uv} = \frac{2Z_u Z_v}{Z+\bar{Z}} \tag{1}$$

The Lax pair associated with equation (1) is

$$\psi_x = M\psi, \qquad \psi_t = N\psi \tag{2}$$

where

$$\begin{aligned} M &= \frac{Z_x}{Z+\bar{Z}} m_1 + \frac{\bar{Z}_x}{Z+\bar{Z}} m_2 \\ N &= \frac{Z_t}{Z+\bar{Z}} n_1 + \frac{\bar{Z}_t}{Z+\bar{Z}} n_2 \end{aligned} \tag{3}$$

along with

$$\begin{aligned} m_1 &= \begin{pmatrix} \frac{1}{2} & K \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad m_2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -K & \frac{1}{2} \end{pmatrix} \\ n_1 &= \begin{pmatrix} \frac{1}{2} & -K^{-1} \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad n_2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ K^{-1} & \frac{1}{2} \end{pmatrix} \end{aligned} \tag{4}$$

K being the eigenvalue.

Let us denote an eigenvector of (2) as $(\psi_1 \psi_2)^t$. We then construct the equations satisfied by the squared eigenfunctions:

$$\sigma = \psi_1^2, \qquad \eta = \psi_2^2, \qquad \chi = \psi_1 \psi_2 \tag{5}$$

which are

$$\begin{aligned} \sigma_x &= u\sigma + 2Kf\chi \\ \eta_x &= -u\eta - 2Kf^*\chi \\ \chi_x &= Kf\eta - Kf^*\sigma \end{aligned} \tag{6}$$

The temporal evolution is governed by

$$\begin{aligned} \sigma_t &= v\sigma - \frac{2g}{K}\chi \\ \eta_t &= -v\sigma + \frac{2g^*}{K}\chi \\ \chi_t &= \frac{g^*\sigma}{\eta} - \frac{g\eta}{K} \end{aligned} \tag{7}$$

where

$$u = f - f^*, \qquad f = \frac{Z_x}{Z + \bar{Z}}$$
$$v = g - g^*, \qquad g = \frac{Z_t}{Z + \bar{Z}} \tag{8}$$

The starting observation for the periodic inverse problem is that

$$(\chi^2 - \eta\sigma)_x = (\chi^2 - \eta\sigma)_t = 0 \tag{9}$$

as a consequence of (6) and (7). So that $\chi^2 - \eta\sigma$ is a constant or integral of motion. Let us denote it by $P(k^2)$ as K is assumed to be independent of t. So that

$$\chi^2 - \eta\sigma = P(K^2) = \sum_{\gamma=1}^{2N+2} P_\gamma K^{2\gamma} \tag{10}$$

where P_γ's are constant of motion. We now seek analytic series solutions to equations (6) and (7) in the form

$$\sigma = \sum_{\gamma=0}^{N} \sigma_\gamma K^{2\gamma+2}, \qquad \eta = \sum_{\gamma=0}^{N} \eta_\gamma K^{2\gamma+2}, \qquad \chi = \sum_{\gamma=0}^{N} \chi_\gamma K^{2\gamma+1} \tag{11}$$

We next substitute the series forms for σ, η, χ in (6), (7), and (10), and equate various powers of K to get the following equations for the coefficient functions:

$$\begin{aligned} \chi_{\gamma t} &= g^*\sigma_\gamma + g\sigma_\gamma^* \\ \sigma_{\gamma-1\,t} &= v\sigma_{\gamma-1} - 2g\chi_\gamma \\ \chi_{\gamma x} &= i(\sigma_N^*\sigma_{\gamma-1} - \sigma_N\sigma_{\gamma-1}^*) \\ \frac{1}{i}\sigma_{\gamma x} &= (\sigma_N + \sigma_N^*)\sigma_\gamma + 2\sigma_N\chi_\gamma \end{aligned} \tag{12}$$

Since right-hand side of equation (10) is a polynomial in K we can assume it to be given in terms of its zeros and hence the same type of representation can also be assumed to be valid for σ. By matching the coefficient of the highest-order term we rewrite σ as

$$\begin{aligned} \sigma &= K^2\sigma_N(K^2 - \mu_1)\ldots(K^2 - \mu_N) \\ &= K^2\sigma_N \prod_{i=1}^{N} (K^2 - \mu_i) \end{aligned} \tag{13}$$

Equation (13) together with (6) and (7) immediately yields

$$\frac{f}{f^*}=\frac{\sigma_N}{\eta_N} \tag{14}$$

and

$$\begin{aligned}\sigma_{N-1}&=-\sigma_N \sum \mu_i\\ \gamma_{N-2}&=\sigma_N \sum_{i<j} \mu_i\mu_j\end{aligned} \tag{15}$$

Now it is not difficult to observe from equations (6) and (7) that χ is real and $\eta=-\sigma^*$. So we also get

$$|\sigma_N|^2=P_{2N+2} \tag{16}$$

and

$$\sigma_N=-\eta_N^*$$

The analog of the scattering data in the case of the periodic inverse problem is really the zeros of σ, that is, μ_i's whose motion in the (χt) plane will yield information about the structure of the nonlinear field Z. By using (13) in (6) and (7) we obtain

$$\frac{\partial\mu_j}{\partial\chi}=\frac{2f[P(\mu_j)]^{1/2}}{(\mu_j)^{1/2}\prod_{\substack{i=1\\ i\neq j}}^{N}(\mu_i-\mu_j)} \tag{17}$$

along with

$$\frac{\partial\mu_j}{\partial t}=\frac{\sum \mu_{it}[P(\mu_j)]^{1/2}}{\mu_j^{3/2}\psi_N\prod_{\substack{i=1\\ i\neq j}}^{N}(\mu_i-\mu_j)} \tag{18}$$

which yields information regarding the variation of the zeros μ_i with respect to χ and t. To explicitly solve for $\partial\mu_i/\partial t$ we observe that the right-hand side of equation (18) may be written as $Q(\mu_i)/\psi_N$, so that

$$\frac{\partial\mu_i/\partial t}{\sum \partial\mu_i/\partial t}=\frac{Q(\mu_i)}{\psi_N}, \qquad \text{where } Q(\mu_i)$$

is some function of μ_i. Now summing over i on both sides of this equation we obtain

$$1 = \frac{\sum Q(\mu_i)}{\psi_N} \qquad \text{or } \psi_N = \sum_{i=1}^{N} Q(\mu_i)$$

$$\frac{\partial\mu_i/\partial t}{\sum \partial\mu_i/\partial t} = \frac{Q(\mu_i)}{\sum Q(\mu_i)}$$

So that we can take for $\partial\mu_i/\partial t$

$$\frac{\partial\mu_i}{\partial t} = Q(\mu_i) = \frac{[P(\mu_i)]^{1/2}}{\mu_i^{3/2} \prod_{\substack{i=1 \\ i\neq j}}^{N} (\mu_i - \mu_j)} \tag{19}$$

Equations (17) and (19) cannot be integrated by ordinary methods but recourse must be taken to the theory of Abelian integrals on Riemann surfaces. For this purpose we define the functions $l_j(\chi t)$ with the help of Abelian differentials as follows. Firstly from equation (14) we infer that the simplest choice for f is

$$f = i\sigma_N \tag{19}$$

which along with (17) gives

$$\frac{\partial\mu_j}{\partial\chi} = \frac{2[P(\mu_j)]^{1/2}}{(\mu_j)^{1/2} \prod_{\substack{i=1 \\ i\neq j}}^{N} (\mu_i - \mu_j)} \tag{20}$$

Now let R be the Reimann surface of the hyperelliptic curve $[ZP(Z)]^{1/2}$. The genus of the surface is N. This Reimann surface is obtained by cross-connecting two copies of the Z plane which are cut along the segments $[-\infty 0][E_0E_1][E_2E_3]\ldots[E_{2N}\infty]$. Let a_j and b_j be the closed contours defined as in the literature then we denote a basis of Abelian different of the first kind by

$$du_j = \sum_{i=0} C_{ji}K^i[KP(K)]^{-1/2}dK \tag{21}$$

Then the l_j's are defined as

$$l_j = -\sum_{\alpha=1}^{N} \int_{\mu_\alpha^0}^{\mu_\alpha} du_j \tag{22}$$

Now we consider the variation of functions $l_j(\chi t)$ we respect to both χ and

t. Differentiating (22) with respect to χ

$$\begin{aligned}\frac{\partial l_j}{\partial \chi}(\chi t) &= -\sum_{\alpha=1}^{N}\sum_{i=0} C_{ij}\mu_\alpha^i[\mu_\alpha P(\mu_\alpha)]^{-1/2}\, d\mu_\alpha \frac{\partial \mu_\alpha}{\cdot\partial \chi}\\ &= -\sum_{\alpha=1}^{N}\sum_{i=0} C_{ji}\mu_\alpha^i[\mu_\alpha P(\mu_\alpha)]^{-1/2}\frac{d\mu_\alpha 2[P(\mu_\alpha)]^{1/2}}{(\mu_\alpha)^{1/2}\prod\limits_{\substack{K=1\\K\neq\alpha}}^{N}(\mu_\alpha-\mu_K)}\\ &= -\sum_{\alpha=1}^{N}\sum_{i=0}\frac{C_{ji}2\mu_\alpha^{i-1}}{\prod\limits_{\substack{K=1\\ \alpha\neq j}}^{N}(\mu_\alpha-\mu_K)} = -2C_j, N \end{aligned} \tag{23}$$

Similarly we get

$$\begin{aligned}\frac{\partial l_j}{\partial t}(\chi t) &= -\sum_{\alpha=1}^{N}\sum_{i=0} C_{ji}\mu_\alpha^i[\mu_\alpha P(\mu_\alpha)]^{-1/2}\frac{\partial \mu_\alpha}{\partial t}\\ &= -\sum_{\alpha=1}^{N}\sum_{i=0}\frac{C_{ji}\mu_\alpha^{i-2}}{\prod\limits_{\substack{K=1\\K\neq\alpha}}^{N}(\mu_\alpha-\mu_K)}\\ &= -\sum_i \frac{C_{ji}}{1}x\delta_{i-2,N-1} = -C_{j,N+1}\end{aligned} \tag{24}$$

So that the transformation through the Abelian differential linearizes the flow on the Reimann surface and hence we can write

$$l_j(\chi, t) = -2C_{j,N}\chi - C_{j,N+1}t + l_0$$

So the problem of determining μ_j is solved by the Abelian inversion of the integrals (22). It is now well known that such determination is explicitly realized by the Reimann θ functions. So finally we observe that the inverse problem solution is complete it. σ_N can be determined through μ_j and lastly f by equation (19). For this since

$$|\sigma_N|^2 = P_{2N+2}$$

We represent σ_N as

$$\sigma_N = (P_{2N+2})^{1/2}e^{i\alpha(\chi t)} \tag{25}$$

and obtain from equations (12)

$$\begin{aligned}\frac{\partial \alpha}{\partial \chi} &= 2(P_{2N+2})^{1/2}\cos\alpha + 2\psi_N\\ \frac{\partial \alpha}{\partial t} &= \frac{(P_{2N+2})^{1/2}}{\psi_N}\sin\alpha\end{aligned} \tag{26}$$

where

$$\psi_N = \sum f(\mu_i)$$

So once α can be determined from equation (26) via μ_i the nonlinear field is obtained through equation (19).

In our above discussions we have pointed out that the methods of periodic inverse problem can be successfully applied for obtaining the conoidal wave solutions of $SO(2,1)\sigma$ model field theory. The linearization of μ_i flows is achieved through the Abel mapping, and finally the nonlinear field is determined through some algebraic identities [equation (19)].

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